Fluid Dynamics Key Equations

Steady Streamlines
For a two-dimensional velocity field:

$\vec{u}(x,y) = u(x,y)\hat{\imath} + v(x,y)\hat{\jmath}$

The streamlines are:

$$\int \frac{1}{v(x,y)}\,dy = \int \frac{1}{u(x,y)}\,dx$$

Unsteady Streamlines & Pathlines
For a two-dimensional unsteady field:

$\vec{u}(x,y,t) = u(x,y,t)\hat{\imath} + v(x,y,t)\hat{\jmath}$

The parametrised pathlines are:

$$x = \int u(x,y,t)\,dt \qquad y = \int v(x,y,t)\,dt$$

Forces in Fluids
Shear Stress, τ

$$\tau = \frac{F}{A}$$

$$\tau = \mu\frac{du}{dy}$$

Pressure Force $F_P = PA$

Viscous Force $F_V = \tau A = \mu\frac{du}{dy}A$

Kinematic Viscosity, v $v = \frac{\mu}{\rho}$

Fluid Statics
Hydrostatic Equation

$$\frac{dP}{dz} = -\rho g$$

$$\Delta P = -\rho g \Delta z$$

Resultant Pressure Force

$$F_R = \int_A P\,dA$$

$$F_R = \int_A \rho g y\,dA$$

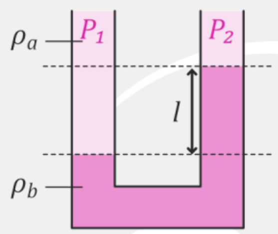
Point of Application

$$y' = \frac{\int_A Py\,dA}{\int_A P\,dA}$$

$$y' = \frac{\int y^2\,dy}{\int y\,dy}$$

- dA must be a function of y

Manometer

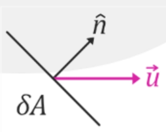


$$P_2 - P_1 = gl(\rho_a - \rho_b)$$

Archimedes' Principle & Buoyancy

"The magnitude of upthrust is equal to the weight of water displaced"

Mass Flow Rate
Vectorial



$$\dot{m} = \int_A \rho\,\vec{u}\,.\,\overrightarrow{dA}$$

Perpendicular $\vec{u}$ $\dot{m} = \rho u A$

Reynold's Transport Theorem

$$\frac{d}{dt}\int_{Vsys(t)} \eta\rho\,dV = \frac{d}{dt}\int_{CV} \eta\rho\,dV + \int_{CS} \eta\rho\,\vec{u}\,.\,\overrightarrow{dA}$$

Rate of change of N in the system	Rate of change of N in the CV and system	Net flow rate of N out of the CV

- N is the property being conserved, $N = \eta m$
- $\rho\eta$ is the property N per unit volume

Conservation of Mass

$$\underbrace{\frac{d}{dt}\int_{CV} \rho \, dV}_{\text{Rate at which the CV gains mass}} = \underbrace{-\int_{CS} \rho \, \vec{u} \, . \, \vec{dA}}_{\text{Net flow rate of mass out of the CV}}$$

Rate at which the CV gains mass

Net flow rate of mass out of the CV

Algebraic Formulation $\quad \dfrac{dm}{dt} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$

Steady Flow

$$\sum \dot{m}_{in} = \sum \dot{m}_{out}$$

$$\int_{CS} \rho \, \vec{u} \, . \, \vec{dA} = 0$$

Steady, uniform flow with constant density $\quad \sum uA_{in} = \sum uA_{out}$

Conservation of Momentum

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \vec{u}\rho \, dV + \int_{CS} \vec{u}\rho \, \vec{u} \, . \, \vec{dA}$$

- Because $N = M = m\vec{u}$, so $\eta = \vec{u}$

Steady Flow

$$\sum \vec{F} = \int_{CS} \vec{u}\rho \, \vec{u} \, . \, \vec{dA}$$

Steady Flow, resolved into components

$$\sum F_x = \int_{CS} u_x \rho \, \vec{u} \, . \, \vec{dA}$$

$$\sum F_y = \int_{CS} v_y \rho \, \vec{u} \, . \, \vec{dA}$$

Steady, uniform, constant density

$$\sum F_x = \sum_{out} \dot{m}u - \sum_{in} \dot{m}u$$

$$\sum F_y = \sum_{out} \dot{m}v - \sum_{in} \dot{m}v$$

With Conservation of Mass for one inlet and outlet

$$\sum F_x = \dot{m}(u_{out} - u_{in})_x$$

$$\sum F_y = \dot{m}(v_{out} - v_{in})_y$$

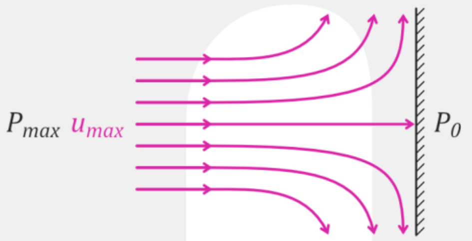
The Bernoulli Equation

$$\frac{P_1}{\rho} + \frac{1}{2}u_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}u_2^2 + gz_2 = c$$

This assumes:

- Steady flow
- Inviscid Flow
- Incompressible (constant ρ) flow
- Two points are on the same streamline/ have the same Bernoulli constant

Stagnation Point Flow



$$\frac{P_{max}}{\rho} + \frac{1}{2}u_{max}^2 = \frac{P_0}{\rho}$$

Conservation of Energy for Steady Flow

$$\dot{Q} - \dot{W} = \int_{CS} \left(\frac{P}{\rho} + \frac{1}{2}u^2 + gz + e\right)\rho \, \vec{u} \, . \, \vec{dA}$$

1st Thermo Law $\quad \dfrac{dE_{sys}}{dt} = \dot{Q} - \dot{W}$

Derived from RTP $\quad \eta = \dfrac{1}{2}u^2 + gz + e$

For a uniform velocity profile:

$$q - w = \Delta\left(\frac{P}{\rho} + \frac{1}{2}u^2 + gz + e\right)$$

The Pipe Flow Energy Equation (PFEE)

$$\frac{P_1}{\rho} + \frac{1}{2}u_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}u_2^2 + gz_2 + w_L - w_P$$

- w_L is the lost energy
- $w_P = w$ is the pump work

$$\frac{P_1}{\rho g} + \frac{1}{2g}u_1^2 + z_1 = \frac{P_2}{\rho g} + \frac{1}{2g}u_2^2 + z_2 + h_L - h_P$$

- h_L is the lost head
- h_P is the pump head

The pipe flow energy equation only applies for flow that is:

- Steady
- Adiabatic
- Incompressible
- Uniform velocity field
- Between a single inlet and outlet

Laminar Flow between Horizontal Plates

$$u(y) = -\frac{h^2 \Delta P}{2\mu L}\left(1 - \frac{y^2}{h^2}\right)$$

Where u_{max} equals $-\frac{h^2 \Delta P}{2\mu L}$

Laminar Flow in a Circular Pipe

$$u(r) = -\frac{R^2 \Delta P}{4\mu L}\left(1 - \frac{r^2}{R^2}\right)$$

Where u_{max} equals $-\frac{R^2 \Delta P}{4\mu L}$

Turbulent Flow in Circular Pipes

Reynold's Number, Re $\quad Re = \frac{\rho u d}{\mu} = \frac{ud}{v}$

Mean Velocity, u $\quad u = \frac{Q}{A}$

Friction Factor, f $\quad f = \frac{Re}{64}$

Relative roughness, r $\quad r = \frac{\varepsilon}{D}$

Lost Head

Lost head, h_L $\quad h_L = h_f + h_l$

Major Losses, h_f $\quad h_f = f\frac{Lu^2}{2dg}$

Minor Losses, h_l $\quad h_l = k\frac{u^2}{2g}$

Pump Head

Pump head, h_P

$$h_P = \frac{w_P}{g}$$

$$h_P = \frac{\dot{W}_P}{\dot{m}g}$$

$$h_P = \frac{\Delta P_P}{\rho g}$$

Material Derivative

For a quantity # following a particle:

$$\frac{D\#}{Dt} = \frac{\partial \#}{\partial t} + u\frac{\partial \#}{\partial x} + v\frac{\partial \#}{\partial y} + w\frac{\partial \#}{\partial z}$$ in Cartesian

$$\frac{D\#}{Dt} = \frac{\partial \#}{\partial t} + grad(\#).\vec{u}$$ in vector form

Eulerian & Lagrangian

The Eulerian system follows a point in space over time: $\vec{x}$

The Lagrangian system follows a particle's position through space and time: $\vec{\chi}$

The Continuity Equation (mass)

$$\frac{\partial \rho}{\partial t} + div(\rho\vec{u}) = 0 \qquad \frac{1}{\rho}\frac{D\rho}{Dt} = -div(\vec{u})$$

Assuming continuum only

Incompressible flow $\quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} = div(\vec{u}) = 0$

Compressibility, β

$$\frac{dV}{V} = \beta\, dP$$

$$\beta = -\frac{1}{V}\frac{\partial V}{\partial P}$$

The Euler Equation

$$\rho \frac{D\vec{u}}{Dt} = -\overrightarrow{grad}(P) + \vec{f}_{body}$$

Assuming continuum & inviscid flow

Body forces $\vec{f}_{body} = \rho\vec{g} + \vec{f}_{ref} + \vec{f}_{ex}$ — $\vec{f}_{ex}$ are applied external forces

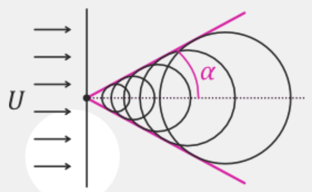
Non-inertial forces $\vec{f}_{ref} = -\rho\vec{a}_{ref}$ — $\vec{a}_{ref}$ is the non-inertial acceleration

Surface pressure force $\overrightarrow{grad}\,(P) = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}$

Speed of Sound

In a perfect gas $c = \sqrt{\gamma R T}$

Mach number, M_n $M_n = \frac{U}{c}$



Mach cone angle, α $\alpha = \sin^{-1}\frac{1}{M_c}$

Isentropic Flow

$$\frac{P_0}{P} = \left[1 + \frac{\gamma - 1}{2} M_n^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{\gamma - 1}{2} M_n^2\right]^{\frac{1}{\gamma-1}}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M_n^2$$

Assuming

- Steady flow
- Inviscid flow
- No body forces
- Perfect gas
- Isentropic process
- Single streamline

Bulk velocity, U_b $U_b(z) = \frac{1}{A}\int_A u_z \, dA$

Area-velocity relation $\frac{dA}{A} = \frac{dU}{U}(M_n^2 - 1)$

Converging-diverging nozzle, throat area A^*:

$$\frac{A}{A^*} = \frac{1}{M_n}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M_n^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Maximum flow rate through a nozzle:

$$\dot{m}_{max} = P_0 A \sqrt{\frac{\gamma}{RT}}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$